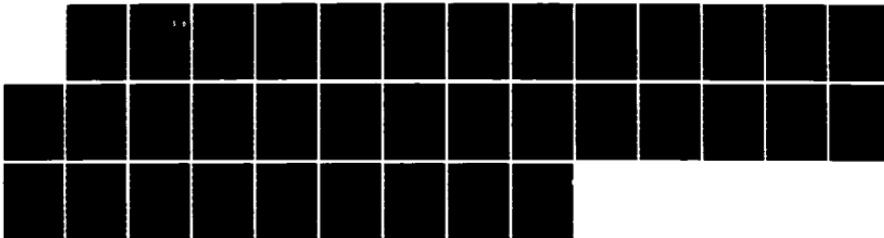
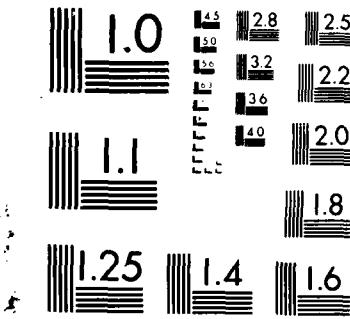


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SOME NEW APPROACHES TO  
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Sept. 1986

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ABSTRACT

In this paper we extend and generalize to the multivariate set-up our earlier investigations related to expected remaining life functions and general hazard measures including representations and stability theorems for arbitrary probability distributions in terms of these concepts. (The univariate case is discussed in detail in Kotz and Shanbhag, Advances in Applied Probability 12 (1980).)

Keywords and Phrases: Expected life function, hazard measure, reliability representation and stability theorems, identifiability, multivariate probability distributions.

AMS 1979 subject classification: 62H05, 62N05.

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## 1. INTRODUCTION

In the earlier paper by the authors, Kotz and Shanbhag (1980) (to be referred to as KSh for brevity) presented a detailed discussion of new approaches to univariate probability distributions. We concentrated on representations and characterizations of probability distribution functions in terms of conditional expectations (specifically in terms of the expected remaining life function - e.r.l. function) and in terms of hazard measures.

In the course of our investigations, we succeeded in extending, generalizing and simplifying a number of results dealing with e.r.l. functions and hazard measures which have appeared in the literature of the last two decades. We also presented some convergence theorems which shed light on the structure of e.r.l. functions, hazard measures and distribution functions in both the continuous and discrete cases (but not restricted to these cases only).

In many instances of practical applications, requiring model building, there are indications of such results being of special potential importance.

The present paper is structured along the lines of KSh (1980) but is an initial attempt towards studying more subtle and difficult problems of multivariate distributions. In this paper, we shall attempt to unify, extend, generalize and simplify results scattered in the literature related to structures of multivariate distributions (in particular but not exclusively of a non-absolutely continuous nature), of various definitions of hazard measures (unlike the univariate case there is no unique definition of this concept in the multivariate case in the literature). Among other results, an over-compassing generalization of the scalar multivariate hazard measure is given and an overall



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structure as well as certain convexity properties and their implications related to this measure are revealed. In addition, we define and investigate multivariate analogues and extensions of e.r.l. functions and trace their relations, first to the multivariate probability distribution functions and then to the corresponding univariate concept on the one hand, as well as to (various generalizations of) multivariate hazard measures on the other. Following the approach adopted in KSh (1980) for the univariate case, we do not restrict ourselves necessarily to non-negative random variables. (The notions of the hazard measure as well as that of the e.r.l. functions in the literature are usually limited to the non-negative case.)

Most of the groundwork as far as the convergence and representation theorems is concerned has been laid in KSh (1980). However, in the present paper we clarify, using examples of specific distributions, some ambiguities and certain inconsistencies related to the structure of various characteristics of multivariate distributions in our search for the most meaningful and practically attractive expressions and representations of these distributions which would expose the hidden dependencies among jointly distributed random variables. These findings could prove to be of some significance to future developments at least in areas such as reliability and pattern recognition.

## 2. A GENERALIZED MULTIVARIATE HAZARD GRADIENT AND A MULTIVARIATE GENERALIZATION OF THE e.r.l. FUNCTION

In this section, we shall give, among other things, two theorems that follow as direct corollaries of KSh (1980). These concern respectively a generalized multivariate hazard gradient and an analogous

multivariate generalization of the e.r.l. function.

For multivariate distributions, there exist in the literature basically two approaches to defining hazard functions, both confined predominantly to absolutely continuous distributions on Euclidean spaces.

The first definition, adopted and analyzed by, among others, Basu (1971) and Puri and Rubin (1973), is a straightforward extension of the univariate concept. (A purely discrete case was also considered by Puri and Rubin (1973).) The hazard function of a random vector  $\underline{x} = (x_1, \dots, x_p)$  is defined in this case to be a real-valued function  $r$  on  $\{\underline{x}: \bar{F}(\underline{x}) > 0\}$  with values

$$r(\underline{x}) = f(\underline{x})/\bar{F}(\underline{x}),$$

where  $\underline{x} = (x_1, \dots, x_p) \in R^p$ ,  $f(\underline{x})$  is the probability density function, and  $\bar{F}(\underline{x})$  is the survivor function given by

$$\bar{F}(\underline{x}) = P(X \geq \underline{x}).$$

(Here as well as in what follows the inequalities for vectors are to be understood componentwise.) This concept was further discussed by Block (1977) where additional closely related variants were proposed, and treated in a somewhat more unified manner in Galambos and Kotz (1978). We intend to generalize this definition and examine it in greater detail. However, since our contribution in this case is to be rather substantial without relying very heavily on KSh (1980), we shall deal with it separately in the next section (i.e., Section 3 of the paper).

The second approach, due to Johnson and Kotz (1975a) and Marshall (1975), defines a multivariate hazard gradient (in an absolutely continuous case) as the vector-valued function  $\underline{h}$  on  $\{\underline{x}: \bar{F}(\underline{x}) > 0\}$  with values

$$\begin{aligned} \underline{h}(\underline{x}) &= \left( -\frac{\partial}{\partial x_1}, \dots, -\frac{\partial}{\partial x_p} \right) \log \bar{F}(\underline{x}) \\ &= -\text{grad } \log \bar{F}(\underline{x}) \end{aligned}$$

(except for a set of Lebesgue measure zero). As was shown by Marshall (1975) in the absolutely continuous case, the vector-valued  $\tilde{h}$  uniquely determines the probability distribution function (d.f.) or equivalently the survivor function. Note that each one of the components of  $\tilde{h}(x)$  depends in general on all the variables  $x_i$  ( $i = 1, 2, \dots, p$ ). In the first part of this section (i.e. in part a) we shall generalize the gradient  $\tilde{h}$  to the case of arbitrary d.f.'s and at the same time reduce some redundancy existing in the structure of the components of this gradient. The main result involving a representation given in this part subsumes Marshall's (1975) result and is essentially a corollary of Propositions 5 and 8 of KSh (1980).

In KSh (1980) - motivated by the remark contained in Shanbhag (1970) and the results of Hamdan (1972), Kotlarski (1972), Shanbhag and Bhaskara Rao (1975) and Gupta (1975) - we also extended the concept of the e.r.l. function of a positive random variable to an arbitrary random variable and have given a representation for a probability distribution in terms of this function. Some possibilities of the applicability of the concept in practice have been indicated in KSh (1980) and the references cited above. (Also, see Hall and Wellner (1981), Hollander and Proschan (1984) and Glanzel et al (1984) for further information and references on the e.r.l. function.) A variety of multivariate generalizations of this function can of course be constructed. However, we intend in this case to deal only with a certain construction that has features closely resembling those of the multivariate hazard function of the present section. The representation theorem in this latter case follows as a corollary of KSh (1980). In view of the prevailing analogy, we shall devote the second part of this

section (i.e. part b) to discussing this particular version of e.r.l. functions and revealing some of its properties including the aformentioned theorem. For a related but independently carried out investigation of multivariate analogues of e.r.l. functions, the reader may wish to consult Zahedi (1985). This work is however along different lines.

a. A generalized hazard gradient and some of its basic properties.

Let  $p \geq 2$ ,  $F$  be a d.f. on  $\mathbb{R}^p$  and  $\underline{x} = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_p)$  be a random vector distributed according to this d.f. Let  $v_F^{(i)}(\cdot | \underline{x}_{(i+1)})$  with  $\underline{x}_{(i)} = (\underline{x}_i, \underline{x}_{i+1}, \dots, \underline{x}_p)$ ,  $\underline{x}_{(1)} = \underline{x}$  denote the hazard measure on  $\mathbb{R}^1$  for the conditional distribution of  $X_i$  given that  $\underline{x}_{i+1} \geq \underline{x}_{i+1}, \dots, \underline{x}_p \geq \underline{x}_p$  (as stipulated in Section 4 of KSh (1980)) for every  $\underline{x}_{(i+1)} \in \mathbb{R}^{p-i}$  and  $i = 1, 2, \dots, p-1$ . (We define the conditional distribution to be arbitrary for any conditioning set of measure zero.) Also, let  $v_F^{(p)}(\cdot)$  denote the corresponding hazard measure on  $\mathbb{R}^1$  for the marginal distribution of  $X_p$ . Extending and modifying the definition of Johnson and Kotz (1975a) and Marshall (1975), we call the family

$$\{v_F^{(i)}(\cdot | \underline{x}_{(i+1)}): \underline{x}_{(i+1)} \in \mathbb{R}^{p-i}, i = 1, 2, \dots, p-1\}, v_F^{(p)}(\cdot)$$

the hazard gradient relative to the d.f.  $F$ . We have the following theorem which is essentially a corollary of Propositions 5 and 8 of KSh (1980) (see, also, Cox (1972)):

THEOREM 1. The survivor function corresponding to  $F$  is represented by

$$\begin{aligned} \bar{F}(\underline{x}) &= P(X=\underline{x}) = \prod_{i=1}^p \left\{ \prod_{y_i \in D_i(\underline{x}_{(i)})} \{1 - v_F^{(i)}(\{y_i\} | \underline{x}_{(i+1)})\} \right\} \\ &\quad \times \exp[-v_F^{(c,i)}((-\infty, \underline{x}_i] | \underline{x}_{(i+1)})], \underline{x} \in \mathbb{R}^p \end{aligned} \quad (2.1)$$

and for a continuous  $F$  the representation is

$$\bar{F}(\underline{x}) = \exp\{-\sum_{i=1}^p v_F^{(i)}((-\infty, x_i] | \underline{x}_{(i+1)})\}, \underline{x} \in R^p, \quad (2.2)$$

where the notation  $v_F^{(p)}(\cdot | \underline{x}_{(p+1)})$  is used for convenience to denote  $v_F^{(p)}(\cdot)$ ,  $e^{-\infty}$  is defined to be zero,  $D_i(x_{(i)})$  is the set of real points  $y_i < x_i$  at which  $v_F^{(i)}(\{y_i\} | \underline{x}_{(i+1)})$  is positive, and  $v_F^{(c,i)}(\cdot | \underline{x}_{(i+1)})$  the continuous (non-atomic) part of  $v_F^{(i)}(\cdot | \underline{x}_{(i+1)})$ . Furthermore, if  $F$  is continuous and  $\{F_n : n=1,2,\dots\}$  is a sequence of d.f.'s on  $R^p$ , then using the same notation

$$v_{F_n}^{(i)}((-\infty, x_i] | \underline{x}_{(i+1)}) \rightarrow v_F^{(i)}((-\infty, x_i] | \underline{x}_{(i+1)}) \quad (2.3)$$

for each  $\underline{x}$  such that  $\bar{F}(\underline{x}) > 0$  and  $i = 1, 2, \dots, p$  if and only if  $\{F_n\}$  converges to  $F$ .

Proof. (2.1) and, if  $F$  is continuous, (2.2) follow immediately from Proposition 5 of KSn (1980) in view of the relation

$$P(\underline{X} \geq \underline{x}) = P(X_p \geq x_p) \prod_{i=1}^{p-1} P(X_i \geq x_i | X_{i+1} \geq x_{i+1}, \dots, X_p \geq x_p), \\ \underline{x} \in R^p. \quad (2.4)$$

If  $F$  is continuous, then the marginal distribution function of  $X_p$  is continuous and for every  $\underline{x}$  such that  $\bar{F}(\underline{x}) > 0$  and  $i = 1, 2, \dots, p-1$ , the conditional distribution of  $X_i$  given  $X_{i+1} \geq x_{i+1}, \dots, X_p \geq x_p$  is continuous. Also, if  $\underline{X}^{(n)} = (X_1^{(n)}, \dots, X_p^{(n)})$  for each  $n \geq 1$  is a random vector distributed according to  $F$ , then for each  $n \geq 1$

$$P(\underline{X}^{(n)} \geq \underline{x}) = P(X_p^{(n)} \geq x_p) \prod_{i=1}^{p-1} P(X_i^{(n)} \geq x_i | X_{i+1}^{(n)} \geq x_{i+1}, \dots, X_p^{(n)} \geq x_p), \\ \underline{x} \in R^p. \quad (2.5)$$

Applying Proposition 8 of KSh (1980) to the survivor functions on the r.h.s. of (2.5), it can be easily verified that the convergence part of the theorem is valid.

Remark 1.

For absolutely continuous distributions, representation (2.2) reduces to that of Marshall (1975). Both (2.2) and (2.1) are thus extensions of Marshall's hazard gradient representation. Moreover, the general representation for purely discrete distributions follows from (2.1) in the obvious manner.

Remark 2.

The "convergence" part of Theorem 1 fails to be valid if the assumption of continuity of  $F$  is omitted. Examples 1-3 presented in KSh (1980) following Proposition 8 in Section 4 are sufficient to illustrate this situation.

Remark 3.

The hazard gradient obviously has other versions when the ordering of the variables is altered. Under a specific situation, one may find a particular version to be the most natural and easiest to handle. In that case, we shall of course consider the corresponding ordering to be the one implied in our Theorem 1. A similar remark applies to the result of Theorem 2 below.

Remark 4.

The following observation related to univariate hazard measures may be appropriate at this point. (See also the beginning of Section 4 of KSh (1980).) If  $G$  is a d.f. on  $\mathbb{R}^1$ , then according to representation

(4.1) in KSh (1980) either

$$\sum_{x_r \in D} (1 - v_G(\{x_r\})) = 0 \text{ or } H_c(\infty) = \infty,$$

where  $v_G$  is the hazard measure corresponding to  $G$ ,  $D$  is the set of discontinuities of  $v_G$  and  $H_c(x) = v_G^{(c)}((-\infty, x])$ ,  $v_G^{(c)}$  being the continuous part of  $v_G$ . Whenever the right extremity of  $G$  is not one of its discontinuity points, we have  $v_G(\{x_r\}) < 1$  for all  $x_r \in D$ . Now the Borel zero-one law and relation (16) given in Burrill (1972), p.245, imply that  $\sum_{x_r \in D} (1 - v_G(\{x_r\})) = 0$  if and only if  $\sum_{x_r \in D} v_G(\{x_r\}) = \infty$  provided  $v_G(\{x_r\}) < 1$ ,  $x_r \in D$ . This leads us to the relation

$$v_G((-\infty, \infty)) = \sum_{x_r \in D} v_G(\{x_r\}) + H_c(\infty) = \infty \quad (2.6)$$

whenever the right extremity of  $G$  is not one of its discontinuity points. (This result was obtained earlier by Shanbhag (1979) using a somewhat different argument.)

Remark 5.

As a corollary of Theorem 1, it follows that the components of  $\underline{x}$  are independent if and only if there exists a version of the hazard gradient of  $F$  such that  $v_F^{(i)}(\cdot | \underline{x}_{(i+1)})$  is independent of  $\underline{x}_{(i+1)}$  for each  $i = 1, 2, \dots, p-1$ . The theorem also yields several other interesting corollaries. In particular, since the theorem also implies that every distribution on  $R^p$  is characterized by its hazard gradient, one could obviously use it to give further characterizations of distributions, such as the Marshall-Olkin bivariate distribution or Fréchet's multivariate distribution with continuous marginals or a multivariate Pareto distribution, for which the hazard gradients are of a particularly appealing form.

b. The generalized e.r.l. function and some relevant comments.

In view of Proposition 3 of KSh (1980), (2.4) in the proof of Theorem 1 above implies that under some mild assumptions there exists a representation for the survivor function of every p-component random vector  $\underline{X} = (\underline{x}_1, \dots, \underline{x}_p)$  in terms of the conditional expectations  $E\{h_i(\underline{x}_i) | \underline{x}_i \geq \underline{x}_i, \dots, \underline{x}_p \geq \underline{x}_p\}$ ,  $(\underline{x}_i, \dots, \underline{x}_p) \in \mathbb{R}^{p-i+1}$  of monotone transforms  $h_i$ ,  $i = 1, 2, \dots, p$ . This is given by the following Theorem 2. The theorem yields, among other things, that if  $\underline{X}$  is a random vector with  $E\{\underline{x}_i^+\} < \infty$  for all  $i = 1, 2, \dots, p$  (where  $\underline{x}_i^+ = \max\{0, \underline{x}_i\}$ ), then the conditional expectations  $E\{\underline{x}_i - \underline{x}_i | \underline{x}_i \geq \underline{x}_i, \dots, \underline{x}_p \geq \underline{x}_p\}$ ,  $i = 1, 2, \dots, p$ ,  $\underline{x} = (\underline{x}_1, \dots, \underline{x}_p) \in \mathbb{R}^p$  (and hence  $E\{\underline{X} - \underline{x} | \underline{X} \geq \underline{x}\}$ ,  $\underline{x} \in \mathbb{R}^p$ ) characterize the distribution of  $\underline{X}$ ; the representation in this latter case is also obvious now. Since the family of expectations  $\{E\{\underline{x}_i - \underline{x}_i | \underline{x}_i \geq \underline{x}_i, \dots, \underline{x}_p \geq \underline{x}_p\} : i = 1, 2, \dots, p, \underline{x} \in \mathbb{R}^p\}$  avoids some of the redundancies existing in the function  $E\{\underline{X} - \underline{x} | \underline{X} \geq \underline{x}\}$ ,  $\underline{x} \in \mathbb{R}^p$  and has all the obvious requirements of an e.r.l. function, it would be reasonable to adopt it to be the e.r.l. function of a multivariate probability distribution on  $\mathbb{R}^p$ .

**THEOREM 2.** (A representation theorem). Let  $\underline{X} = (\underline{x}_1, \dots, \underline{x}_p)$  be a random vector with p components and  $h_i$ ,  $i = 1, 2, \dots, p$  be real-valued non-decreasing functions on the real line such that  $E\{h_i^+(\underline{x}_i)\} < \infty$  for all  $i = 1, 2, \dots, p$  (where  $h_i^+(\underline{x}_i) = \max\{0, h_i(\underline{x}_i)\}$ ). If  $h_i$ ,  $i = 1, 2, \dots, p$ , are such that  $h_i(\underline{x}_i) < E\{h_i(\underline{x}_i) | \underline{x}_i > \underline{x}_i, \underline{x}_{i+1} \geq \underline{x}_{i+1}, \dots, \underline{x}_p \geq \underline{x}_p\}$  whenever  $P\{\underline{x}_i > \underline{x}_i, \underline{x}_{i+1} \geq \underline{x}_{i+1}, \dots, \underline{x}_p \geq \underline{x}_p\} > 0$ , then the survivor function corresponding to  $\underline{X}$  is given by

$$P(\underline{X} \geq \underline{x}) = \prod_{i=1}^p G_i(x_{(i)}), \quad \underline{x} = (x_1, \dots, x_p) \in \mathbb{R}^p, \quad (2.7)$$

where  $\underline{x}_{(i)} = (x_1, \dots, x_p)$

$$\text{and } G_i(x_{(i)}) = \begin{cases} 0 & \text{if } x_i > b_i^* \text{ for } b_i^* < \infty \\ \lim_{\substack{y_i \uparrow b_i}} G_i(y_i, \underline{x}_{(i+1)}) & \text{if } x_i = b_i^* \text{ for } b_i^* < \infty \\ \lim_{\substack{y_i \rightarrow -\infty}} \frac{g_i(y_i, \underline{x}_{(i+1)})}{g_i(\underline{x}_{(i)})} \{ \prod_{\substack{z \in D^{(i)} \\ y_i < z < x_i}} g_i^*(z, \underline{x}_{(i+1)}) \} \exp \left\{ - \int_{y_i}^{x_i} \frac{dh_i^{(c)}(z)}{g_i(z, \underline{x}_{(i+1})} \right\} & \text{if } x_i > b_i^*, \end{cases} \quad (2.8)$$

in which  $D^{(i)}_{y_i, x_i}$  denotes the set of discontinuity points of  $h_i$  in  $[y_i, h_i]$ ,  $h_i^{(c)}$  denotes the continuous part of  $h_i$  (i.e. of its right continuous version),

$$\begin{aligned} g_i(x_{(i)}) &= E\{h_i(x_i) | \underline{x}_{(i)} \geq \underline{x}_{(i)}\} - h_i(x_i^-), \\ g_i^*(z, \underline{x}_{(i+1)}) &= \left\{ \frac{g_i(z, \underline{x}_{(i+1)}) - (h_i(z) - h_i(z^-))}{g_i(z^+, \underline{x}_{(i+1)}) + (h_i(z^+) - h_i(z))} \right\} \frac{g_i(z^+, \underline{x}_{(i+1)})}{g_i(z, \underline{x}_{(i+1)})}, \end{aligned} \quad (2.9)$$

$$\text{and } b_i^* = \begin{cases} \infty & \text{if } \{y: \lim_{x_i \uparrow y} E\{h_i(x_i) | \underline{x}_{(i)} \geq \underline{x}_{(i)}\} \text{ exists and } \leq h_i(y)\} \text{ is empty} \\ \inf\{y: \lim_{x_i \uparrow y} E\{h_i(x_i) | \underline{x}_{(i)} \geq \underline{x}_{(i)}\} \text{ exists and } \leq h_i(y)\} & \text{otherwise} \end{cases}$$

with  $\underline{x}_{(i)} = (x_i, \dots, x_p)$ .

(The conditional distributions are defined arbitrarily when the conditioning sets are of measure zero; also (2.8) and (2.9) in the statement above are to be read without  $\underline{x}_{(i+1)}$  in the case of  $i = p$ .)

Remark 6.

In view of Theorem 3 and the information given in the Remarks in Section 3 of KSh (1980), it is possible to present several extensions and variants of Theorem 2 given above.

Remark 7.

If  $h_i$ 's in Theorem 2 are assumed additionally to be continuous, then the representation (2.7) with  $G_i$ 's given by (2.8) without the term  $\{ \prod_{\substack{z \in D \\ y_i x_i}} g_i^*(z, \underline{x}_{(i+1)}) \}$  and with  $h_i^{(c)}$ 's replaced by  $h_i$ 's is valid.

Remark 8.

If  $h_i$  ( $i = 1, 2, \dots, p$ ) of Theorem 2 are taken as strictly increasing, the representation (2.7) for a survivor function is obviously valid in the case of every distribution satisfying the integrability condition of the theorem. One may be interested in seeing whether there exists a representation for the survivor function for  $X$  in terms of the conditional expectations corresponding to a fewer number of functions, which are appealing in some sense, at least when the domains of the definition of  $h_i$  are taken as Euclidean spaces with  $h_i(X_i)$  considered above replaced by  $h_i(\underline{x}^{(i)})$ ,  $\underline{x}^{(i)}$  being a subvector of  $\underline{x}$ . However, it is not difficult to see that in general merely with the integrability condition such a representation does not exist. This could be verified by noting, for example, that if  $h_i$ ,  $i = 1, 2, \dots, p-1$  are given to be real-valued Borel

measurable functions on  $\mathbb{R}^p$ , then there exist random vectors  $X$  and  $Y$  with distinct distributions having a common support (such as  $\{(0, \dots, 0), (1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ ) such that

$$E\{h_i(X)|X \geq x\} = E\{h_i(Y)|Y \geq x\} \text{ for all } x \in \mathbb{R}^p \text{ and } i = 1, 2, \dots, p-1.$$

Remark 9.

Prakasa Rao (1974) has essentially attempted to solve under some constraints the problem mentioned in Remark 8. He has given in this context a uniqueness theorem in the bivariate case under certain assumptions. The following example shows that the theorem is not valid.

EXAMPLE 1. Define  $h$  to be a real-valued function on  $\mathbb{R}^2$  such that

$$h(x,y) = (1-e^{-x^2})\xi(y), \quad x, y \in \mathbb{R},$$

where

$$\xi(y) = \begin{cases} c & \text{if } y \leq 1 \\ c + \frac{(y-1)^3}{6} & \text{if } 1 < y \leq 2 \\ c + (y-2) + \frac{(3-y)^3}{6} & \text{if } 2 < y \leq 4 \\ c + 2 + \frac{(y-5)^3}{6} & \text{if } 4 < y \leq 5 \\ c + 2 & \text{if } y > 5, \end{cases}$$

where  $c$  is a positive number. Alternatively, one could consider the  $h$  with a slightly more trivial situation of  $\xi \equiv c$  for  $c \neq 0$ . Let  $(X, Y)$  and  $(Z, W)$  be random vectors with absolutely continuous independent non-negative components such that  $X$  and  $Z$  are identically distributed but the distributions of  $Y$  and  $W$  are not identical. Also assume the random vectors to be such that their marginal distributions have all left extremities to be equal to zero and

$$P(Y \leq 1) = P(W \leq 1), \quad P(Y \leq y|Y \geq 1) = P(W \leq y|W \geq 1) \text{ for all } y \geq 1.$$

Observe that all the assumptions in Theorem 2.1 of Prakasa Rao (1974) are satisfied with  $x_0 = y_0 = 0$ . Moreover  $(X, Y)$  and  $(Z, W)$  satisfy Prakasa Rao's stipulation (2.0). However, in this case, the conclusions of the theorem are not valid. (It is obviously possible to illustrate this point by other examples of a similar nature.)

Remark 10.

In view of Theorem 2, characterizations based on e.r.l. functions are now obvious for the well known distributions such as the Marshall-Olkin bivariate distribution, the Farlie-Gumbel-Morgenstern distribution discussed in Johnson and Kotz (1975b), Gumbel's bivariate exponential distribution, the multivariate Pareto distribution and several other multivariate distributions appearing in Johnson and Kotz (1972). One could also apply the theorem to arrive at further characterizations based on conditional expectations for distributions such as Fréchet's and those discussed by Krishnaiah (1977). The following example may serve as an illustration of this point.

EXAMPLE 2. (Fréchet's bivariate continuous distribution).

Consider  $F$  to be the continuous d.f. on  $\mathbb{R}^2$  such that the corresponding survivor function is given by

$$\bar{F}(x_1, x_2) = \min\{1-F_1(x_1), 1-F_2(x_2)\}, (x_1, x_2) \in \mathbb{R}^2$$

with  $F_1$  and  $F_2$  as univariate d.f.'s. Clearly, since  $F$  is assumed to be continuous, we require  $F_1$  and  $F_2$  to be continuous here also. Define

$$h_i(x_i) = (F_i(x_i))^{\alpha_i}, x_i \in \mathbb{R}^1, i = 1, 2,$$

where  $0 < \alpha_i < \infty$  and fixed. Then it follows that if  $\underline{x} = (x_1, x_2)$  is a random vector with d.f.  $F$ , we have for every  $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$  and  $i = 1, 2$

$$\begin{aligned} E\{h_i(x_i)|x_i \geq x\} \\ = \begin{cases} \frac{1}{1+\alpha_i} \{1-(G(x))^{1+\alpha_i}\} \{1-G(x)\}^{-1} & \text{if } G(x) < 1 \\ 1 & \text{if } G(x) = 1, \end{cases} \end{aligned}$$

where  $G(x) = \max\{F_1(x_1), F_2(x_2)\}$ . (On the set  $\{G(x) = 1\}$ , one could also define  $E\{h_i(x_i)|x_i \geq x\}$  differently.) Obviously, given  $\alpha_i$  and  $F_i$ ,  $\{E(h_1(x_1)|x_1 \geq x_1, x_2 \geq x_2), E(h_2(x_2)|x_2 \geq x_2): (x_1, x_2) \in R^2\}$  characterizes the distribution considered above among all bivariate distributions which have the right extremities of the marginals lying in (left extr.  $F_1$ , right extr.  $F_2$ ) and (left extr.  $F_2$ , right extr.  $F_1$ ) respectively. (This distribution has several other interesting characterization properties also, the recent characterization based on discretized Shannon entropy given in Bertoluzza and Forte (1985) being one of these.)

### 3. EXTENDED VERSIONS OF THE RESULTS OF BASU AND PURI AND RUBIN DEALING WITH THE HAZARD FUNCTION

We shall now discuss a rather substantial generalization of what is known in the literature as the "scalar" multivariate hazard function.

Let, as in the previous section,  $F$  be a d.f. on  $R^p$ ,  $X$  be a  $p$ -component random vector with this distribution and  $F$  be the corresponding survivor function. Denote by  $P_F$  the measure determined by  $F$  on (the Borel  $\sigma$ -field  $B_p$  of)  $R^p$ . Since, in the multivariate case, we can have an  $F$  such that  $P_F\{\tilde{x}: \bar{F}(\tilde{x}) = 0\} > 0$ , (e.g., if we take  $F$  to be continuous such that  $P_F\{\tilde{x} = (x_1, \dots, x_n): x_i = -x_2\} = 1$ , we obtain  $P_F\{\tilde{x}: \bar{F}(\tilde{x}) = 0\} = 1$ ), the definition of a hazard measure in KSh (1980) is not extendable as it stands. However, if we restrict ourselves only to the set  $C$  (say) of distributions  $F$  for which  $\bar{F}(\cdot) > 0$  almost surely [ $P_F$ ], the definition

in KSh (1980) of a hazard measure admits an obvious extension. Suppose then that  $F \in \mathbf{C}$  and define  $v_F$  to be the scalar hazard measure on  $\mathbb{R}^P$  given by

$$v_F(B) = \int_B \frac{1}{\bar{F}(x)} dP_F(x) \text{ for all } B \in \mathcal{B}_P. \quad (3.1)$$

The integral on the r.h.s. of the equation can be written following the accepted convention in the literature as  $\int_B \frac{1}{\bar{F}(x)} dF(x)$ .

In the case when  $F$  is an absolutely continuous d.f. with respect to the Lebesgue measure on  $\mathbb{R}^P$ ,  $v_F$  also possesses this property and thus the Radon-Nikodym derivative becomes the hazard function, studied by earlier authors, a.e. on  $\{x: \bar{F}(x) > 0\}$ . It follows from the investigations of Basu (1971) and Puri and Rubin (1973) (see also, Galambos and Kotz (1978)) that the measure  $v_F$  does not in general determine uniquely the distribution  $F$ . Consider then  $\mathcal{D}_F$  to be the set of d.f.'s on  $\mathbb{R}^P$  that are members of  $\mathbf{C}$  having the same scalar hazard measure as  $F$ . Clearly the set  $\mathcal{D}_F$  defined herein is convex although not necessarily closed relative to weak convergence.

Consider now the set of all d.f.'s on the compactified Euclidean space  $[-\infty, \infty]^P$ . There exists a normed linear space of which this is a compact subset and the corresponding relative metric is a metric of weak convergence. Then, as a further subset of this compact set, the closure  $\bar{\mathcal{D}}_F$  of the set  $\mathcal{D}_F$  is also compact. (For simplicity we abuse the notation slightly here and elsewhere in this section by denoting the set of all d.f.'s on  $[-\infty, \infty]^P$  which are extensions of members of  $\mathcal{D}_F$  also by  $\mathcal{D}_F$ .) Since  $\bar{\mathcal{D}}_F$  is also convex, Choquet's theorem

(cf. Phelps (1965) page 19 and also Kendall (1963)) implies that each  $F^* \in \mathcal{D}_F$  can be represented as the centroid or barycenter of a probability measure on the Borel  $\sigma$ -field of the linear space which is concentrated on the set of extreme points of  $\bar{\mathcal{D}}_F$ . In general, the problem of obtaining the extreme points of  $\bar{\mathcal{D}}_F$  or merely of  $\mathcal{D}_F$  seems to be a difficult one and we have not as yet obtained any positive information in this connection. However, through a theorem and two corollaries to follow, we shall provide some valuable information concerning the problem of characterizing  $F$  on the basis of  $v_F$ . This gives, among other things, the Poisson-Martin representation for  $F$  in terms of  $v_F$  when  $F$  is continuous and a more natural extension of the univariate hazard measure to the multivariate case than the hazard gradient of the last section, possessing the uniqueness and stability requirements.

Before discussing our main results of this section, the following instructive examples making some specific points are worth revealing.

EXAMPLE 3. Let

$$F(x) = \prod_{i=1}^p F_i(x_i), \quad x = (x_1, \dots, x_p) \in R^p,$$

where  $F_i$  are continuous d.f.'s on  $R^1$ . Then, appealing to the result of Puri and Rubin (1973) or our observation above concerning a representation for the members of  $\bar{\mathcal{D}}_F$ , we can easily see that each member  $F^* \in \mathcal{D}_F$  has the following form:

$$F^*(x) = \int_{R^p} \prod_{i=1}^p \{1 - (1 - F_i(x_i))^{\lambda_i}\} dG(\lambda), \quad x \in R^p, \quad (3.2)$$

where  $G$  is a d.f. on  $R^p$  such that the corresponding measure is concen-

trated on the set  $\{\lambda: \lambda_i > 0, i = 1, 2, \dots, p, \prod_{i=1}^p \lambda_i = 1\}$ . Also, this can be seen via the Poisson-Martin integral representation given for the members of  $\mathcal{D}_F$  in Corollary 1 below. Incidentally, in the present case, the extreme points of  $\mathcal{D}_F$  are given precisely by the d.f.'s  $F^*$  of the form

$$F^*(x) = \prod_{i=1}^p (1 - F_i(x_i))^{\lambda_i}, \quad x \in \mathbb{R}^p$$

with  $\lambda_i > 0, i = 1, 2, \dots, p$  and  $\prod_{i=1}^p \lambda_i = 1$  and any extreme point of  $\mathcal{D}_F$  is either an extreme point of  $\mathcal{D}_F$  or a d.f. on  $[\infty, \infty]^p$  which is the weak limit of a sequence of extreme points of  $\mathcal{D}_F$ . Looking at an arbitrary member  $F^*$  given by (3.2) in the case of  $p \geq 2$  for  $\mathcal{D}_F$ , we observe a curious property of  $\mathcal{D}_F$  that if  $F^* \in \mathcal{D}_F$  and any  $p-1$  of the  $p$  univariate marginals of  $F^*$  agree with those corresponding to  $F$ , then  $F^* = F$ . In other words, we have in this case that if a d.f. on  $\mathbb{R}^p$  has  $p-1$  of its univariate marginals precisely the same as those corresponding to  $F$  and its scalar hazard measure on  $\mathbb{R}^p$  is defined and is given by  $v_F$ , then this d.f. has to be  $F$ . Since every univariate d.f. is uniquely determined by its hazard measure, we could also restate this property utilizing only hazard measures. (For some recent advances connected with the results discussed herein, see Lau and Rao (1982), Rao and Shanbhag (1984) and Davies and Shanbhag (1984).)

EXAMPLE 4. Let  $p \geq 2$ ,  $k$  be a real number and  $S$  be a countable subset of  $\mathbb{R}^{p-1}$ . Also let  $G$  denote the set of d.f.'s on  $\mathbb{R}^{p-1}$  that are concentrated on  $S$  giving a positive probability mass to each point of  $S$ . For each  $G \in G$ , let  $F_G$  denote a d.f. on  $\mathbb{R}^p$  such that it is concentrated on  $\{x: x \in \mathbb{R}^p, \sum_1^p x_i = k\}$  with

$$F_G(x_1, \dots, x_{p-1}, \infty) = G(x_1, \dots, x_{p-1}), (x_1, \dots, x_{p-1}) \in R^{p-1}$$

(in the usual notation). It is easily seen that here  $v_F$  are all (well defined and) identical. If we now consider  $p \geq 4$  and any of the  $F_G$ 's to be  $F$ , then it is clearly seen that the condition that  $F^* \in \mathcal{D}_F$  does not imply  $F^* = F$  even if it is given that  $F^*$  has all of its univariate marginals or bivariate marginals to be the same as those of  $F$ . However, for the  $F$  in this example, the condition that  $F^* \in \mathcal{D}_F$  together with

$$F^*(x_1, \dots, x_{p-1}, \infty) = F(x_1, \dots, x_{p-1}, \infty), (x_1, \dots, x_{p-1}) \in R^{p-1}$$

implies that  $F^* = F$ . Note also that here we have the set of extreme points of  $\mathcal{D}_F$  to be empty and the set of extreme points of  $\bar{\mathcal{D}}_F$  to be the closure (relative to weak convergence) of the set of the degenerate d.f.'s on  $[-\infty, \infty]^p$  that are concentrated on  $\{x: x \in R^p, \sum_i^p x_i = k\}$ ; clearly here the situation of the last example that each  $F^* \in \mathcal{D}_F$  has an integral representation in terms of the extreme points of  $\mathcal{D}_F$  is not valid.

In spite of certain isolated cases, such as that of Fréchet's distribution of Example 2 or of a d.f.  $F$  that satisfies for some  $b \in R^p$  the conditions  $F(b) = 1$  and  $\bar{F}(b) = P_F(\{b\}) > 0$ , in which the  $F$  is characterized by  $v_F$ , it now follows that, in general, unless at least one of the  $(p-1)$ -variate marginals of the distribution (or something equivalent to it) is given,  $v_F$  does not characterize  $F$ . One might then be interested to know whether  $F$  is characterized by  $v_F$  given any one of the  $(p-1)$ -variate marginals. Our attempt to answer this question has been only partially successful so far and the findings of this investigation are presented, among other things, in the following results.

We are now ready to give our main theorem of the section together with the two of its interesting corollaries. (The reader can find some analogy between the proof of the theorem given here and Seneta's (1984)

proof of the Poisson-Martin integral representation theorem for a super regular vector corresponding to a non-negative matrix.)

THEOREM 3. If  $F^* \in \mathcal{D}_F$  and, for each  $i = 1, 2, \dots, p$ , we have (in the standard notation)

$$F^*(x_1, \dots, x_{i-1}, \infty, x_{i+1}, \dots, x_p) = F(x_1, \dots, x_{i-1}, \infty, x_{i+1}, \dots, x_p) \\ \text{for all } x_i \in \mathbb{R}^1, i = 1, 2, \dots, i-1, i+1, \dots, p, \quad (3.3)$$

then  $F^* = F$ . Furthermore, given an  $F^* \in \bar{\mathcal{D}}_F$ , there exists a probability measure  $\mu^*$  on the set of all d.f.'s,  $G$  on  $[-\infty, \infty]^p$ , such that

$$F^*(x) = \int_K G(x) d\mu^*(G), \quad x \in [-\infty, \infty]^p, \quad (3.4)$$

where  $\mu^*(K) = 1$  and  $K$  is the closure (relative weak convergence) of the set of the d.f.'s  $K_t(\cdot)$  for  $t$  such that  $F(t), \bar{F}(t) > 0$  ( $\bar{F}$  being the survivor function of  $F$  as in the last section), where each of the  $K_t(\cdot)$  is defined to be a d.f. on  $[-\infty, \infty]^p$  such that it is the degenerate d.f. at  $t$  if  $P_F(\{t\}) = \bar{F}(t)$  and the d.f. satisfying the following otherwise:

$$K_t(x) = \frac{k(x, t)}{\tilde{k}(\tilde{t}, \tilde{t})}, \quad x \in (-\infty, t] \quad (3.5)$$

with

$$k(y, t) = \Delta_t(y) + \sum_{n=1}^{\infty} \int_{(-\infty, y]} \int_{[y_1, t]} \dots \int_{[y_{n-1}, t]} dv_F(y_n) \dots dv_F(y_1), \quad y \in [-\infty, t],$$

$\Delta_t(\cdot)$  being the d.f. degenerate at  $t$ . (The proof of the theorem asserts that  $K_t(\cdot)$  is well defined.)

Proof. In view of Fubini's Theorem and relation  $P_F(B) = \int_B \bar{F}(x) dv_F(x)$

$$= \xi_n(t) + (-1)^{(n+1)p} \int_{(-\infty, t)} \int_{(-\infty, x_n)} \cdots \int_{(-\infty, x_2)} P_F^*((-\infty, x_1)) \\ dv_F(x_1) \cdots dv_F(x_n), \quad n \geq 1, \quad (3.7)$$

where the sequence  $\{\xi_m(t): m = 0, 1, \dots\}$  (for each given  $t$ ) is such that it depends only on  $v_F$  and d.f.'s  $F^*(x_1, \dots, x_{i-1}, \infty, x_{i+1}, \dots, x_p)$ ,  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p) \in R^{p-1}$ ,  $i = 1, \dots, p$ . It follows trivially from (3.6) that the multiple integral on the r.h.s. of (3.7) tends to zero and  $n \rightarrow \infty$ . This in turn implies that the sequence  $\{\xi_n(t): n = 1, 2, \dots\}$  in (3.7) converges to  $\bar{F}^*(t)$  and hence we have that if (3.3) is valid, then

$$\bar{F}^*(t) = \bar{F}(t) \text{ for each } t \text{ such that } \bar{F}(t) > P_F(\{t\}). \quad (3.8)$$

In view of the left continuity of  $\bar{F}$  and  $\bar{F}^*$  and the fact that  $\{x: x \in R^p, \bar{F}(x) = 0\} = \{x: x \in R^p, \bar{F}^*(x) = 0\}$ , we can conclude that if (3.8) is valid, then we have  $\bar{F}^* = \bar{F}$  or equivalently  $F^* = F$ . This establishes the first part of the theorem.

To establish the second part of the theorem, define

$$B = \{t: t \in R^p, F(t), \bar{F}(t) > 0\},$$

$$B_0 = \{t: t \in R^p, \bar{F}(t) = P_F(\{t\}) > 0\},$$

$$\text{and } B_m = \{t: t \in B, \bar{F}(t) \geq P_F(\{t\}) + \frac{1}{m}\}, \quad m = 1, 2, \dots.$$

If  $F^* \in D_F$ , then by the monotone convergence theorem, we get

$$\begin{aligned} \bar{F}^*(x) &= \int_{[x, \infty)} \bar{F}^*(y) dv_F(y) \\ &= \int_{B_0} \bar{K}_t(x) dP_F(t) + \lim_{m \rightarrow \infty} \int_{B_m} \bar{\Delta}_t(x) \bar{F}^*(t) dv_F(t), \quad x \in R^p. \end{aligned} \quad (3.9)$$

Now, for every  $m \geq 1$  and  $\tilde{t} \in B_m$ ,  $\alpha_{\tilde{t}}$  of (3.6) is bounded by  $m/(m+1)$  and hence it follows from (3.6) that  $k(\tilde{x}, \tilde{t})$  is bounded on  $B_m$  for each  $m \geq 1$ . Also, if we define  $\bar{k}(\tilde{x}, \tilde{t}) = k(\tilde{t}, \tilde{t}) - k(\tilde{x}, \tilde{t})$  for each  $\tilde{t} \in B \cap B_0^c$  and  $\tilde{x} \in R^P$  (with this to be zero if  $\tilde{x} \notin (-\infty, \tilde{t}]$ ). Fubini's Theorem implies that for each  $m \geq 1$  and  $\tilde{x} \in R^P$

$$\begin{aligned} & \int_{B_m} \bar{k}(\tilde{x}, \tilde{t}) \bar{F}^*(\tilde{t}) d\nu_{\tilde{F}}(\tilde{t}) \\ &= \int_{B_m} \bar{\Delta}_{\tilde{t}}(\tilde{x}) \bar{F}^*(\tilde{t}) d\nu_{\tilde{F}}(\tilde{t}) + \int_{B_m} \bar{k}(\tilde{x}, \tilde{t}) P_{\tilde{F}}^*(B_m(\tilde{t})) d\nu_{\tilde{F}}(\tilde{t}), \end{aligned} \quad (3.10)$$

where  $B_m(\tilde{t}) = [\tilde{t}, \infty) \cap B_m$ . Observe that (3.10) follows easily from the relations:

$$\begin{aligned} \bar{k}(\tilde{x}, \tilde{t}) &= \bar{\Delta}_{\tilde{t}}(\tilde{x}) + \sum_{n=1}^{\infty} \int_{[\tilde{x}, \tilde{t}]} \int_{[\tilde{x}, \tilde{y}_n]} \dots \int_{[\tilde{x}, \tilde{y}_2]} d\nu_{\tilde{F}}(\tilde{y}_1) \dots d\nu_{\tilde{F}}(\tilde{y}_n), \\ & \quad \tilde{x} \leq \tilde{t}, \quad \tilde{t} \in B \cap B_0^c, \end{aligned}$$

and

$$\int_{B_m(\tilde{y}_n)} \bar{F}^*(\tilde{t}) d\nu_{\tilde{F}}(\tilde{t}) = P_{\tilde{F}}^*(B_m(\tilde{y}_n)), \quad y_n \in B \cap B_0^c, \quad m, n \geq 1.$$

From (3.9) and (3.10), it consequently follows that there exists a sequence  $\{\mu_m : m = 1, 2, \dots\}$  of measures on  $R^P$  such that  $\mu_m(R^P) \leq 1$  for all  $m$  and

$$\bar{F}^*(\tilde{x}) = \lim_{m \rightarrow \infty} \int_B \bar{k}(\tilde{x}, \tilde{t}) d\mu_m(\tilde{t}), \quad \tilde{x} \in R^P, \quad (3.11)$$

which in turn implies that  $\{\mu_m(B)\}$  converges to 1 and hence that there exists a sequence  $\{\mu_m\}$  of probability measures on  $R^P$  for which (3.11) is valid. Since  $K$  is compact, using Parthasarathy's (1967) Theorem 6.4,

it can then be easily seen that there exists a probability measure  $\mu^*$  on  $K$  such that

$$\bar{F}^*(x) = \int_K \bar{G}(x) d\mu^*(G), \quad x \in R^P.$$

Since  $\bar{\mathcal{D}}_F$  is the closure of  $\mathcal{D}_F$ , a further application of Parthasarathy's Theorem yields the validity of the second part of our theorem.

The following two corollaries of Theorem 3 are easy to prove:

**COROLLARY 1.** (The Poisson-Martin representation): If  $F$  is continuous, then we have a d.f.  $F^*$  on  $R^P$  to be a member of  $\mathcal{D}_F$  if and only if it has a representation

$$F^*(x) = \int_{K \cap \mathcal{D}_F} G(x) d\mu(G), \quad x \in R^P,$$

for some probability measure  $\mu$  on  $K \cap \mathcal{D}_F$ . (In the present case, we also have  $K \cap \mathcal{D}_F$  to be a  $G_\delta$  set of the space of all d.f.'s on  $[-\infty, \infty]^P$ .)

**COROLLARY 2.** The hazard measure  $v_F$  jointly with the hazard measures relative to all the univariate and multivariate marginals of  $F$  determines  $F$  uniquely. (This corollary can be verified by induction.)

#### Remark 11.

It can be noted that the result of Corollary 1 does not remain valid if the assumption that  $F$  is continuous is dropped. Also, in view of what we have observed, it can be concluded that if  $F$  is continuous, then we have the set of extreme points of  $\mathcal{D}_F$  to be a nonempty subset of  $K$  and each  $F^* \in \mathcal{D}_F$  to be the barycenter of a probability measure that is carried by the set of extreme points of  $\mathcal{D}_F$ .

Remark 12.

The finite collection of hazard measures given in Corollary 2 appears, in spite of the restriction that  $F \in \mathbf{C}$ , to be a more natural multivariate analog of the univariate hazard measure than the hazard gradient of the last section. A stability theorem for the collection is valid when  $F$  is continuous, as is shown by Corollary 3 of the next section.

## 4. A STABILITY THEOREM

We conclude the paper by proving and commenting in this section on a general stability theorem for probability measures on metric spaces, which yields, among other things, the two stability propositions in KSh (1980) as simple corollaries. The proof of the present theorem uses Prohorov's (1956) and related theorems in Billingsley (1968) dealing with the convergence of probability measures. It might be instructive to compare this with the proofs of earlier stability propositions in KSh (1980). The techniques used for proving the theorem here are indeed of a more global nature than those which are sufficient in the case of probability measures on the real line.

Now, let  $S$  be a metric space,  $T$  an index set,  $\mathbf{S}$  the Borel  $\sigma$ -field on  $S$ ,  $\mathbf{P}$ ,  $\mathbf{P}_1$ ,  $\mathbf{P}_2$ ,  $\mathbf{P}_3$  families of probability measures on  $(S, \mathbf{S})$ ,  $\{\mathbf{A}_t : t \in T\}$  a family of collections of sets with  $\mathbf{A}_t \subseteq \mathbf{S}$  for every  $t \in T$ , and  $\{h(\cdot | t, A_t, P) : A_t \in \mathbf{A}_t, P \in \mathbf{P}, t \in T\}$  a family of real-valued Borel measurable functions on  $(S, \mathbf{S})$  satisfying the following conditions in which the notation  $D(t, A_t, P)$  stands for the set of discontinuity points of  $h(\cdot | t, A_t, P)$ .

(i)  $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \subset \mathbf{P}$ , also  $P_2$  is closed (under weak convergence),

(ii)  $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, \dots \in \mathbf{P}_1$  and  $\{P_n^{(1)} : n \geq 1\}$  converges weakly to  $P^* \in \mathbf{P} = h(\cdot | t, A_t, P_n^{(1)}) \rightarrow h(\cdot | t, A_t, P^*)$  as  $n \rightarrow \infty$  uniformly almost surely [ $P^*$ ] on  $A_t \cap D^c(t, A_t, P^*)$  and

$$\sup_{n \geq 1} E_{P_n^{(1)}} \{ |h(\cdot | t, A_t, P_n^{(1)})| I_{A_t \cap \{|h(\cdot | t, A_t, P_n^{(1)})| \geq \alpha\}} \} \rightarrow 0$$

as  $\alpha \rightarrow \infty$  for each  $t \in T$  and  $P^*$ -continuity set  $A_t$  in  $A_t$  with  $P^*(A_t)$  positive,

(iii)  $P_1^{(2)}, P_2^{(2)} \in \mathbf{P}_2$  and are distinct  $\Rightarrow$  there exist  $t \in T$  and  $A_t \in \mathbf{A}_t$  such that  $P_1^{(2)}(A_t), P_2^{(2)}(A_t)$  are both positive,  $A_t$  is both  $P_1^{(2)}$ -continuity set and  $P_2^{(2)}$ -continuity set and

$$E_{P_1^{(2)}} \{ h(\cdot | t, A_t, P_1^{(2)}) | A_t \} \neq E_{P_2^{(2)}} \{ h(\cdot | t, A_t, P_2^{(2)}) | A_t \},$$

and

(iv)  $P^{(3)} \in \mathbf{P}_3 \Rightarrow D(t, A_t, P^{(3)})$  has zero  $P^{(3)}$ -measure for every  $t$  in  $T$  and  $P^{(3)}$ -continuity set  $A_t$  in  $A_t$ .

Further, let  $P \in \mathbf{P}$  and  $\{P_n : n \geq 1\}$  be a sequence of members of  $\mathbf{P}_1$  such that  $\{P_n : n = 1, 2, \dots\}$  is relatively compact. Then we have the following stability theorem:

**THEOREM 4.** (a) The condition that

$$P \in \mathbf{P}_3, \{P_n : n \geq 1\} \text{ converges weakly to } P \quad (4.1)$$

implies that

$$E_{P_n} \{ h(\cdot | t, A_t, P_n) | A_t \} \Rightarrow E_P \{ h(\cdot | t, A_t, P) | A_t \} \quad (4.2)$$

as  $n \rightarrow \infty$  for every  $t \in T$  and  $P$ -continuity set  $A_t \in \mathbf{A}_t$  with  $P(A_t) > 0$ .

Moreover, (b) if additionally  $P, P_1, P_2, \dots \in \mathbf{P}_2$  and the set of cluster points of  $\{P_n : n = 1, 2, \dots\}$  (relative to weak convergence) is a subset

of  $\mathbf{P}_3$ , then the converse assertion is valid.

Proof. Assume first that (4.1) is valid. Since  $P \in \mathbf{P}_3$ , it is obvious that the set of discontinuity points of  $h(\cdot | t, A_t, P) I_{A_t}$  has zero  $P$ -measure for every  $t \in T$  and  $P$ -continuity set  $A_t \in \mathbf{A}_t$ . Now, let  $t \in T$  and  $P$ -continuity set  $A_t \in \mathbf{A}_t$  be arbitrarily fixed. Since  $P_n \in \mathbf{P}_1$ ,  $n \geq 1$ , the requirements of Billingsley's (1968) Theorem 5.5 are clearly met with  $h(\cdot | t, A_t, P) I_{A_t}$  as  $h$  and  $h(\cdot | t, A_t, P_n) I_{A_t}$  as  $h_n$ . This theorem implies that  $\{P_n h_n^{-1}, n = 1, 2, \dots\}$  converges weakly to  $P h^{-1}$ . If we now consider  $X_n$ ,  $n \geq 1$  and  $X$  to be some random variables having distributions  $P_n h_n^{-1}$ ,  $n \geq 1$  and  $P h^{-1}$  respectively, we have  $\{X_n : n = 1, 2, \dots\}$  converging to  $X$  in distribution. Also, the fact that  $P_n \in \mathbf{P}_1$ ,  $n \geq 1$  implies that  $\{X_n : n = 1, 2, \dots\}$  considered here is uniformly integrable. Since Billingsley's (1968) Theorem 5.4 yields that  $E\{X_n\} \rightarrow E\{X\}$  as  $n \rightarrow \infty$  in such a situation, we can conclude that

$$E_{P_n} \{h(\cdot | t, A_t, P_n) I_{A_t}\} \rightarrow E_P \{h(\cdot | t, A_t, P) I_{A_t}\} \text{ as } n \rightarrow \infty. \quad (4.3)$$

In view of the assumptions that  $\{P_n\}$  converges weakly to  $P$  and  $A_t$  is a  $P$ -continuity set, it follows that  $P_n(A_t) \rightarrow P(A_t)$  as  $n \rightarrow \infty$ . If  $P(A_t) \neq 0$ , we have (4.2) then as an obvious consequence of (4.3). Hence we have the first part of the stability theorem to be valid.

To establish that the second part of the theorem holds, assume that  $P, P_1, P_2, \dots \in \mathbf{P}_2$  and the set of cluster points of  $\{P_n : n = 1, 2, \dots\}$  is a subset of  $\mathbf{P}_3$  and also that (4.2) is valid. Since each cluster point of  $\{P_n : n = 1, 2, \dots\}$  is an element of  $\mathbf{P}_3$  and  $\{P_n : n = 1, 2, \dots\}$  is relatively compact, we should have a subsequence  $\{P_{n_r} : r = 1, 2, \dots\}$  of  $\{P_n : n = 1, 2, \dots\}$  converging weakly to  $Q \in \mathbf{P}_3$  with  $Q \neq P$  unless (4.1) is

valid. If  $Q^*$  denotes the (weak) limit of a subsequence of  $\{P_n\}$ , then clearly we have  $Q^* \in \mathcal{P}_3$  and hence the first part of the theorem and the validity of (4.2) lead us to

$$E_P\{h(\cdot | t, A_t, P) | A_t\} = E_{Q^*}\{h(\cdot | t, A_t, Q^*) | A_t\} \quad (4.4)$$

for every  $t \in T$  and  $A_t \in \mathcal{A}_t$  such that  $A_t$  is a  $P$ -continuity set with  $P(A_t) > 0$  as well as a  $Q^*$ -continuity set with  $Q^*(A_t) > 0$ . We have assumed that  $P \in \mathcal{P}_2$  and for each  $n \geq 1$ ,  $P_n \in \mathcal{P}_2$  and also we have  $\mathcal{P}_2$  to be closed. In that case, we have  $P, Q^* \in \mathcal{P}_2$  and hence, in view of (4.4),  $Q^* = P$ . It is therefore impossible that (4.1) will not be valid. Hence we have the second part of the theorem.

#### Remark 13.

In the case of  $h(\cdot | t, A_t, P)$  being independent of  $P$ , obviously the part of condition (ii) that  $h(\cdot | t, A_t, P_n^{(1)}) \rightarrow h(\cdot | t, A_t, P^*)$  uniformly almost surely [ $P^*$ ] on  $A_t \cap D^c(t, A_t, P^*)$  for every  $t \in T$  and  $P^*$ -continuity set  $A_t$  with  $P^*(A_t) > 0$  is trivially met. Also, if  $h(\cdot | t, A_t, P)$  are all continuous, then the condition (iv) above is obviously satisfied with  $\mathcal{P}_3 = \mathcal{P}$ . If  $S$  is a Polish space or in particular, if it is a Euclidean space, we have a sequence  $\{P_n : n = 1, 2, \dots\}$  of members of  $\mathcal{P}$  to be relatively compact if and only if it is tight in the sense of Billingsley (1968: p.37) (cf. Theorems 6.1 and 6.2 in Billingsley (1968)). Thus, it is evident that in various specialized situations, the theorem given above has simplified and perhaps more appealing versions.

#### Remark 14.

If the stipulation "the set of cluster points of  $\{P_n : n = 1, 2, \dots\}$  is a subset of  $\mathcal{P}_3$ " is replaced by the weaker stipulation "the set of

cluster points of the range of  $\{P_n : n = 1, 2, \dots\}$  is a subset of  $P_3''$ , Theorem 4 still remains valid provided we also replace "the converse assertion is valid" by "(4.2) implies that  $\{P_n : n = 1, 2, \dots\}$  converges weakly to  $P''$ .

Remark 15.

To illustrate that the stability theorem just proved does not remain valid if the assumptions  $P \in P_3$  and the set of cluster points of  $\{P_n : n = 1, 2, \dots\}$  is a subset of  $P_3$  respectively appearing in the two parts of the theorem are omitted, it is sufficient to consider the following example:

EXAMPLE 5. Let  $\{x_n : n = 1, 2, \dots\}$  be a sequence of strictly increasing, real numbers converging to a real number  $x'$ . Let  $x''$  be a real number greater than  $x'$ . Define  $P$ ,  $P'$ ,  $\{P_n : n = 1, 2, \dots\}$  to be a sequence of probability measures on the Borel  $\sigma$ -field of  $\mathbb{R}^1$  such that for some  $0 < \alpha < 1$

$$P_n(\{x\}) = \begin{cases} \alpha & \text{if } x = x_n \\ 1 - \alpha & \text{if } x = x'', \end{cases}$$

$$P(\{x\}) = \begin{cases} \alpha & \text{if } x = x' \\ 1 - \alpha & \text{if } x = x'', \end{cases}$$

and

$$P'(\{x\}) = \begin{cases} \alpha + \frac{\alpha(d-c)}{x''-x'} & \text{if } x = x' \\ 1 - \alpha - \frac{\alpha(d-c)}{x''-x'} & \text{if } x = x'', \end{cases}$$

where  $c$  and  $d$  are given real numbers such that  $c < d$  and  $\{\alpha(d-c)/(x''-x')\} < 1 - \alpha$ . Also, define  $h$  on  $\mathbb{R}$  such that

$$h(x) = \begin{cases} c & \text{if } x < x' \\ d + (x-x') & \text{if } x \geq x'. \end{cases}$$

If we take  $T = \text{the singleton } \{1\}$ ,  $A_1 = \{(-\infty, x) : -\infty < x < x'\}$ ,  $\mathcal{P} = \{P, P', P_1, P_2, \dots\}$  and  $h(\cdot)|_{A, P^*} = h(\cdot)$  for every member  $A$  of  $A_1$  and  $P^* \in \mathcal{P}$ , then it follows that  $\mathcal{P}$  itself satisfies the requirement of  $P_1$  and  $P_2$  mentioned above. However, in this case we cannot have a nonempty subset  $\mathcal{P}_3$  of  $\mathcal{P}$  satisfying the condition (iv) as required. Consequently, it follows that in this example neither the requirement of  $P \in \mathcal{P}_3$  nor the requirement of the set of cluster points of  $\{P_n : n = 1, 2, \dots\}$  being a subset of  $\mathcal{P}_3$  are met. Observe that here  $\{P_n : n = 1, 2, \dots\}$  converges to  $P$  weakly,  $P \neq P'$  and (4.2) is not valid (since  $E_{P_n}\{h(\cdot)|A\} \neq E_P\{h(\cdot)|A\}$  whenever  $A = (-\infty, x)$  with  $x < x'$ ) but (4.2) with  $P$  replaced by  $P'$  is valid. This implies that with the deletions mentioned above neither the first part of the theorem nor the second part remains valid.

Theorem 4 has several interesting corollaries. In particular it yields that if a characteristic property exists, based on conditional expectations of the type  $E_p\{h(\cdot|t)|A_t\}$  for probability measures  $P$  within a certain class, then, under certain mild conditions, one can produce a stability version of the property. It is easily seen that Proposition 4 of KSh (1980) is an obvious corollary of Theorem 4 and also it is not difficult now to state a stability version of our Theorem 2 of Section 2 based on Theorem 4. (Note that in view of what was revealed in Remark 13, the statement of Theorem 4 simplifies under the situation in Theorem 2.) It is also worth pointing out in passing that (in view of Proposition 5 of KSh (1980)) the "only if" part of Proposition 8 of KSh (1980) follows as a corollary of the first part of Theorem 4 by letting  $S = \mathbb{R}^1$ ,  $T = (-\infty, b)$ ,  $A_t = \{\mathbb{R}^1\}$  for every  $t \in (-\infty, b)$ ,  $\mathcal{P} = \mathcal{P}_1 = \text{the set of measures in the sequence}$

$$\{P_{F_n} : n \geq 0, F_0 = F\}, (P_2 =) P_3 = \{P_F\}$$

and for each  $t$  in  $T$  and  $P^*$  in  $P$

$$h(x|t, A_t, P^*) = \begin{cases} (P^*([x, \infty)))^{-1} I_{(-\infty, t]} & \text{if } P''([x, \infty)) > 0 \\ 0 & \text{otherwise;} \end{cases}$$

moreover, if some simple a priori observations are made and  $P, P_1, P_2$  and  $P_3$  are appropriately redefined, the "if" part of Proposition 8 of KSh (1980) follows from the second part of Theorem 4. Essentially the same argument leads to the following stability version of the characterization result in our Corollary 2 of Section 3. This result clearly subsumes Proposition 8 of KSh (1980).

COROLLARY 3. Let  $p \geq 1$  and  $\{F_n : n = 1, 2, \dots\}$  be a sequence of d.f.'s on  $R^p$  and  $F$  be a continuous d.f. on  $R^p$ . Assume that  $F$  and for each  $n$ ,  $F_n$  are members of the set  $\mathbf{C}$  defined in the last section. Then

$$F_n(x) \rightarrow F(x) \text{ for all } x \in R^p$$

if and only if

$$v_{F_n}^*(x) \rightarrow v_F^*(x) \text{ for all } x \text{ with } \bar{F}(x) > 0,$$

where the notation  $v_G^*(x)$  stands for the vector whose elements (given in some specified order) are  $v_G((-\infty, x])$  and its counterparts relative to all the univariate and multivariate marginals of  $G$ , with appropriate subvectors in place of  $x$  and appropriate number of components in  $\infty$ , and  $\bar{F}$  stands for the survivor function corresponding to  $F$  as in the earlier sections.

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#### ACKNOWLEDGMENTS

The authors acknowledge the NATO Travel Grant No. 1732 which covered the traveling expenses connected with Dr. D.N. Shanbhag's visit to the U.S.A. in July-August, 1980 during which time the initial version of this paper was written and Dr. S. Kotz's visit in Sheffield in May, 1979 during which this paper was initiated. Dr. Kotz's research was also supported by U.S. Office of Naval Research Contract N00014-84-K-0301.

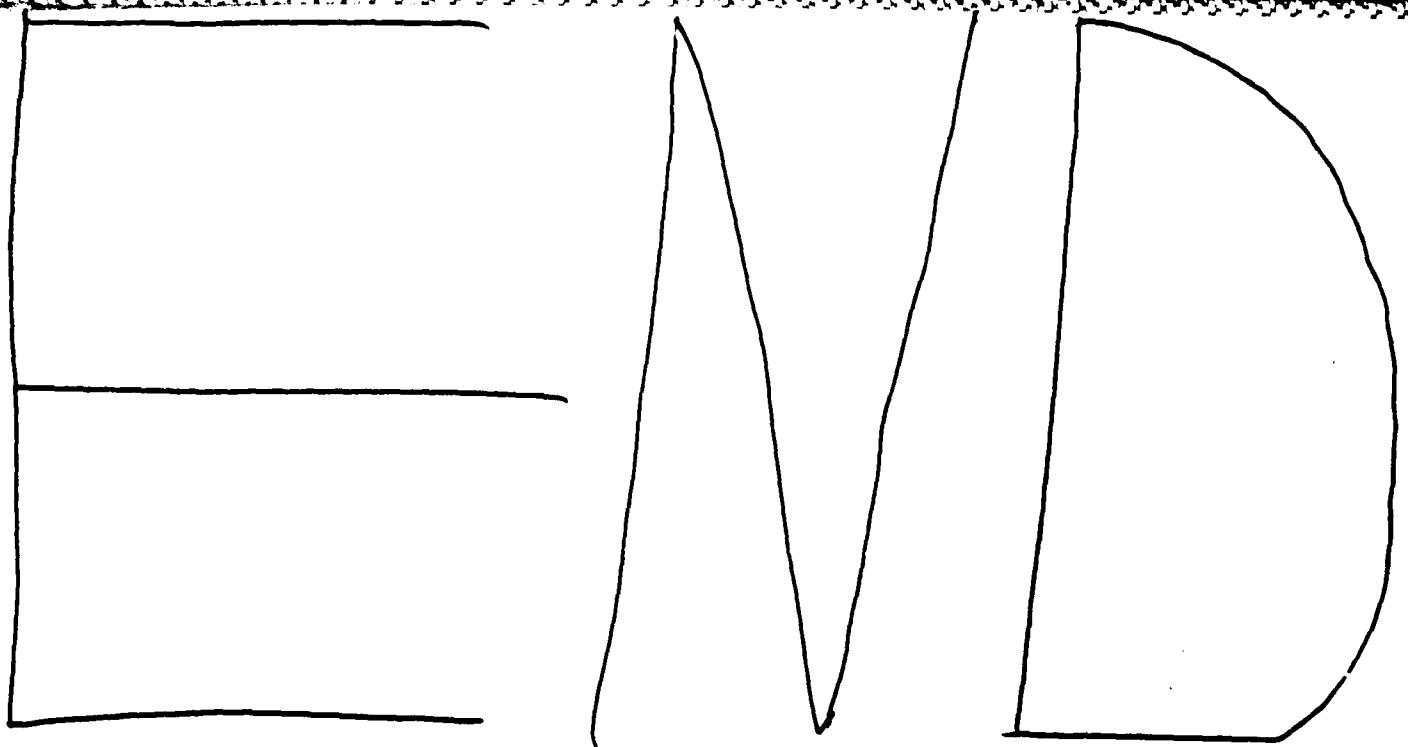
Thanks are due to Dr. William Harkness, Head, Department of Statistics, Penn State University for the arrangements which facilitated the early stages of this research, to Dr. D.G. Kabe, of St. Mary's University, Halifax for pointing out the error in the theorem of Prakasa Rao (1974) and to Dr. P.R. Krishnaiah for his valuable comments on the initial draft, his monumental patience and enthusiastic encouragement.

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SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION <b>UNCLASSIFIED</b>		1b RESTRICTIVE MARKINGS	
2a SECURITY CLASSIFICATION AUTHORITY		3 DISTRIBUTION/AVAILABILITY OF REPORT <b>DISTRIBUTION STATEMENT A</b> Approved for public release; Distribution Unlimited	
2b DECLASSIFICATION/DOWNGRADING SCHEDULE			
4 PERFORMING ORGANIZATION REPORT NUMBER(S)		5 MONITORING ORGANIZATION REPORT NUMBER(S)	
6a NAME OF PERFORMING ORGANIZATION University of Maryland	6b OFFICE SYMBOL (If applicable)	7a NAME OF MONITORING ORGANIZATION Office of Naval Research	
6c ADDRESS (City, State, and ZIP Code) Dep't. of Management and Statistics University of Maryland College Park, Md. 20742		7b ADDRESS (City, State, and ZIP Code)	
8a NAME OF FUNDING/SPONSORING ORGANIZATION ONR	8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER N00014-84-K-0301	
8c ADDRESS (City, State, and ZIP Code) Stat. and Probability Program Office of Naval Research Arlington, Va. 22217		10 SOURCE OF FUNDING NUMBERS	
		PROGRAM ELEMENT NO	PROJECT NO.
		TASK NO	WORK UNIT ACCESSION NO
11 TITLE (Include Security Classification) Some New Approaches to Multivariate Probability Distributions			
12 PERSONAL AUTHOR(S) Shanbhag, D. N. and Kotz, Samuel.			
13a TYPE OF REPORT technical	13b TIME COVERED FROM 9/85 TO 9/86	14. DATE OF REPORT (Year, Month, Day) Sept 1986	15 PAGE COUNT 34 pgs.
16 SUPPLEMENTARY NOTATION			
17 COSATI CODES		18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number) multivariate hazard gradient; expected remaining life functions; stability theorems; integral representations	
FIELD	GROUP	SUB-GROUP	
19 ABSTRACT (Continue on reverse if necessary and identify by block number) In this paper we extend and generalize to the multivariate set-up our earlier investigations related to expected remaining life functions and general hazard measures including representations and stability theorems for arbitrary probability distributions in terms of these concepts. (The univariate case is discussed in detail in Kotz and Shanbhag: <u>Advances in Probability</u> 12 (1980).)			
20 DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION <b>UNCLASSIFIED</b>	
22a NAME OF RESPONSIBLE INDIVIDUAL Samuel Kotz		22b TELEPHONE NUMBER (Area Code) <b>(301) 454-6805</b>	22c OFFICE SYMBOL



2 - 86

